On the Dynamics of Two-Dimensional Array Beam Scanning via Coupled Oscillators

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Abstract - **Arrays of voltage-controlled oscillators coupled to nearest neighbors have been proposed as a means of controlling the aperture phase of one and two-dimensional phased array antennas. It has been demonstrated, both theoretically and experimentally, that one may achieve linear distributions of phase across a linear array aperture by tuning the end oscillators of the array away from the ensemble frequency of a mutually injection-locked array of oscillators. These linear distributions result in steering of the radiated beam. It is demonstrated theoretically here that one may achieve similar beam steering in two dimensions by appropriately tuning the perimeter oscillators of a two-dimensional array. The analysis is based on a continuum representation of the phase in which a continuous function, satisfying a partial differential equation of diffusion type, passes through the phase of each oscillator as its independent variables pass through integer values indexing the oscillators. Solutions of the partial differential equation for the phase function exhibit the dynamic behavior of the array during the beam steering transient.**

injection locking
phased arrays
spatial power combining

I. INTRODUCTION

Some years ago, Liao and York [1] proposed an approach to phased array beam steering which requires no phase shifters. This involves the use of a linear array of voltage-controlled electronic oscillators coupled to nearest neighbors. The oscillators are mutually injection locked by controlling their coupling and tuning appropriately. It was shown that the ensemble frequency of the array is equal to the average of the free running frequencies of the oscillators. When all of the free running frequencies are equal, the oscillators will oscillate in phase which each other if the coupling phase is chosen correctly.[2] Thus, if radiating elements are connected to each oscillator and spaced uniformly along a line, they will radiate a beam normal to the line. The key result is that, if the free running frequencies of the end oscillators of the array are antisymmetrically detuned away from the ensemble frequency by applying appropriate voltages to their tuning ports, a linear phase distribution is established and the radiated beam is steered away from the normal to the array. Only the end oscillators need be detuned and the steering angle is dependent on the amount by which they are detuned.

Recently, Pogorzelski and York presented a formulation which facilitates theoretical analysis of the above beam steering technique.[3][4] It is based on a continuum model in which the oscillator phases are represented in terms of a continuous function satisfying a partial differential equation of diffusion type. This function depends both on time and on an independent variable which, upon taking on integer values, indexes the oscillators. The phase function takes on the value of the phase of a given oscillator when the independent variable takes on the integer value identifying that oscillator and varies smoothly between integer values of the variable. The diffusion equation governing the phase function can be solved via the Laplace transform and the resulting solution exhibits the dynamic behavior of the array as the beam is steered.

The above beam-steering technique can be generalized to two-dimensional arrays in which the beam control voltages are applied to the oscillators on the perimeter of the array. In this paper a continuum model for the two-dimensional case is developed and the dynamic solution for the corresponding aperture phase function as well as the behavior of the resulting far-zone radiation pattern are presented.

II. THE CONTINUUM MODEL IN TWO DIMENSIONS

Consider a 2M+1 by 2N+1 rectangular array of coupled voltage-controlled oscillators. By applying Adler's theory of the dynamics of injection locking [5], it can be shown that the dynamic behavior of such an array is governed by a system of simultaneous differential equations which are first order in time. Specifically, these governing equations are

$$\frac{d\theta_{ij}}{dt} = \omega_{tune,ij} - \sum_{m=-M}^{M} \sum_{n=-N}^{N} \Delta\omega_{lock;ij,mn} \sin(\Phi_{ij,mn} + \theta_{ij} - \theta_{mn}), \tag{1}$$

where $\omega_{tune,ij}$ is the free running frequency of oscillator ij, $\Phi_{ij,mn}$ is the phase associated with the coupling between oscillators ij and mn in the array, and $\Delta\omega_{lock;ij,mn}$ is the locking range associated with that coupling and is given by

$$\Delta\omega_{lock,ij,mn} = \frac{\varepsilon_{ij,mn}\omega_{tune,ij}}{2Q} \frac{\alpha_{mn}}{\alpha_{ij}}, \tag{2}$$

where α_{ij} is the amplitude of the output signal of the ij^{th} oscillator, $\varepsilon_{ij,mn}$ sets the strength of the coupling, and Q is the quality factor of the oscillators. The phase, θ_{ij}, is the phase of the ij^{th} oscillator; that is,

$$\theta_{ij} = \omega_{ref} t + \phi_{ij} , \tag{3}$$

where ω_{ref} is the reference frequency for defining the phase, ϕ_{ij}, of each oscillator. If the oscillators are coupled only to nearest neighbors, equation (1) simplifies to

$$\frac{d\phi_{ij}}{dt} = \omega_{tune,ij} - \omega_{ref} - \sum_{\substack{m=i-1 \\ m\neq i}}^{i+1} \sum_{\substack{n=j-1 \\ n\neq j}}^{j+1} \Delta\omega_{lock;ij,mn} \sin(\Phi_{ij,mn} + \phi_{ij} - \phi_{mn}) . \tag{4}$$

Taking the coupling phase to be zero and assuming that the phase differences between adjacent oscillators are small, the sine functions may be replaced by their arguments. Then, following the reasoning previously used in the one dimensional case [4], it is noted that, if all of the locking ranges are equal, equation (4) becomes

$$\begin{aligned} \frac{d\phi_{ij}}{dt} &= \omega_{tune,ij} - \omega_{ref} - \Delta\omega_{lock}\left[(\phi_{ij} - \phi_{i-1,j-1}) + (\phi_{ij} - \phi_{i-1,j+1}) + (\phi_{ij} - \phi_{i+1,j-1}) + (\phi_{ij} - \phi_{i+1,j+1})\right] \\ &= \omega_{tune,ij} - \omega_{ref} + \Delta\omega_{lock}\left[\phi_{i-1,j-1} + \phi_{i-1,j+1} + \phi_{i+1,j-1} + \phi_{i+1,j+1} - 4\phi_{ij}\right] . \end{aligned} \tag{5}$$

Now, the quantity in brackets on the right side of (5) is seen to be a discrete approximation to the Laplacian operator in two dimensions. Indexing the oscillators by integer values of continuous variables x and y, and representing the oscillator phases by the continuous function $\phi(x,y;\tau)$, one arrives at the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{\partial\phi}{\partial\tau} = -\frac{\omega_{tune} - \omega_{ref}}{\Delta\omega_{lock}} \tag{6}$$

for the phase function, $\phi(x,y;\tau)$, where τ is a dimensionless time measured in inverse locking ranges; that is,

$$\tau = \Delta\omega_{lock}\, t . \tag{7}$$

A unit cell one unit square is associated with each oscillator so that the array extent is given by

$$\begin{aligned} -\left(M + \frac{1}{2}\right) \leq x \leq \left(M + \frac{1}{2}\right) \quad , \\ -\left(N + \frac{1}{2}\right) \leq y \leq \left(N + \frac{1}{2}\right) \quad . \end{aligned} \tag{8}$$

To derive the boundary conditions on the phase function, we employ a generalization of an artifice previously presented in connection with the one dimensional case. [4] That is, an additional set of oscillators is appended on the perimeter of the array so that the augmented array contains (2M+3)(2N+3) oscillators as shown in Figure 1. Note that the added oscillators are not coupled to each other. These fictitious additional oscillators, shown in dashed lines in the figure, are now assumed to be dynamically tuned (i.e., as a function of time) to frequencies such that the phase of each fictitious oscillator is maintained identical with that of its nearest neighbor in the original array. This causes the injection effect between these pairs of oscillators to be identically zero and thus simulates the absence of the fictitious oscillator of each pair. Since the phase difference between each fictitious oscillator and its nearest real neighbor is zero, the spatial derivative of the phase given by this difference is also zero. It is thus easily seen that the desired boundary condition is of Neumann type and may be applied at points halfway between the fictitious oscillators and their nearest real neighbors. Therefore, study of the dynamic behavior of the array is now a matter of solving equation (6) subject to Neumann boundary conditions at $x = \pm\left(a + \frac{1}{2}\right)$ and at $y = \pm\left(b + \frac{1}{2}\right)$ where a and b correspond to index values i=M and j=N denoting the perimeter oscillators in the array.

The ensemble frequency at which the entire mutually injection locked array oscillates can be ascertained by averaging equation (6) over the area of the array. This leads to

$$< \frac{\partial^2 \phi}{\partial x^2} > + < \frac{\partial^2 \phi}{\partial y^2} > - \frac{\partial < \phi >}{\partial \tau} = - \frac{< \omega_{tune} > - \omega_{ref}}{\Delta \omega_{lock}}. \tag{9}$$

The first two terms on the left side are zero by virtue of the Neumann conditions on the boundaries of the array. By definition, the frequency, ω, is related to the phase by

$$\frac{\omega}{\Delta \omega_{lock}} = \frac{d\phi}{d\tau} + \frac{\omega_{ref}}{\Delta \omega_{lock}}. \tag{10}$$

Averaging (10) and substituting into (9) leads to

$$< \omega > = < \omega_{tune} > \tag{11}$$

which is to say, the average oscillation frequency of the oscillators is equal to the average of their free running frequencies. Moreover, since in steady state all of the oscillators oscillate at the same frequency by virtue of their mutual injection locked status, the steady state ensemble frequency of the array as a whole will be equal to this average value of the free running frequencies.

III. THE TWO-DIMENSIONAL RECTANGULAR ARRAY

Consider a rectangular array with (2M+1)(2N+1) oscillators extending over the range

$$-\left(a+\frac{1}{2}\right) \le x \le \left(a+\frac{1}{2}\right) \quad ,$$
$$-\left(b+\frac{1}{2}\right) \le y \le \left(b+\frac{1}{2}\right) \quad . \tag{12}$$

Suppose that the oscillator located at (x′,y′) is detuned at time zero by C locking ranges where C is less than one. The source term on the right side of (6) is then

$$-\frac{\omega_{tune}-\omega_{ref}}{\Delta\omega_{lock}} = -Cu(\tau)\delta(x-x')\delta(y-y'). \tag{13}$$

Substituting (13) into (6) and performing a Laplace transformation with respect to τ yields the partial differential equation,

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}-sg=-\frac{C}{s}\delta(x-x')\delta(y-y'), \tag{14}$$

where $g(x,y)$ is the Laplace transform of $\phi(x,y;\tau)$ with respect to τ and $\phi(x,y;0^+)$ is assumed to be zero; that is, it is assumed that the oscillators are initially in phase with each other. Note that the function g is the Green's function for the transformed differential equation (14) and can be expressed in a straightforward manner in terms of the eigenfunctions of the differential operator satisfying Neumann conditions on the boundary. These eigenfunctions divide naturally into four categories depending upon their symmetry properties. They are

$$\begin{aligned}
f_{ee,mn} &= \frac{1}{N_{ee,mn}}\cosh\left(\sqrt{s_m}x\right)\cosh\left(\sqrt{s_n}y\right) \quad ,\\
f_{oo,k\ell} &= \frac{1}{N_{oo,k\ell}}\sinh\left(\sqrt{s_k}x\right)\sinh\left(\sqrt{s_\ell}y\right) \quad ,\\
f_{ee,m\ell} &= \frac{1}{N_{eo,m\ell}}\cosh\left(\sqrt{s_m}x\right)\sinh\left(\sqrt{s_\ell}y\right) \quad ,\\
f_{oo,kn} &= \frac{1}{N_{oe,kn}}\sinh\left(\sqrt{s_k}x\right)\cosh\left(\sqrt{s_n}y\right) \quad ,
\end{aligned} \tag{15}$$

where the N's normalize the functions and the boundary conditions imply that

$$s_k = -\left(\frac{(2k+1)\pi}{2a+1}\right)^2 \quad , \quad s_m = -\left(\frac{2m\pi}{2a+1}\right)^2 ,$$
$$s_\ell = -\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2 \quad , \quad s_n = -\left(\frac{2n\pi}{2b+1}\right)^2 , \tag{16}$$

for integers, k,ℓ,m,n ranging from zero to infinity. Integrating the squares of the eigenfunctions over the area of the array provides the values of the normalization constants. They are

$$N_{ee,mn} = \frac{1}{2}\sqrt{(2a+1)(2b+1)\varepsilon_m\varepsilon_n} ,$$
$$N_{oo,k\ell} = \frac{1}{2}\sqrt{(2a+1)(2b+1)} ,$$
$$N_{eo,m\ell} = \frac{1}{2}\sqrt{(2a+1)(2b+1)\varepsilon_m} , \tag{17}$$
$$N_{oe,kn} = \frac{1}{2}\sqrt{(2a+1)(2b+1)\varepsilon_n} ,$$

where

$$\varepsilon_m = \begin{matrix} 2; & m=0 \\ 1; & m \neq 0 \end{matrix} \tag{18}$$

is the well known Neumann factor. Thus, the normalized eigenfunctions are

$$\tilde{f}_{ee,mn} = \frac{1}{\sqrt{(2a+1)(2b+1)\varepsilon_m\varepsilon_n}} \cos\left(\frac{2m\pi x}{2a+1}\right) \cos\left(\frac{2n\pi y}{2b+1}\right) ,$$
$$\tilde{f}_{oo,k\ell} = \frac{1}{\sqrt{(2a+1)(2b+1)}} \sin\left(\frac{(2k+1)\pi x}{2a+1}\right) \sin\left(\frac{(2\ell+1)\pi y}{2b+1}\right) ,$$
$$\tilde{f}_{eo,m\ell} = \frac{1}{\sqrt{(2a+1)(2b+1)\varepsilon_m}} \cos\left(\frac{2m\pi x}{2a+1}\right) \sin\left(\frac{(2\ell+1)\pi y}{2b+1}\right) , \tag{19}$$
$$\tilde{f}_{oe,kn} = \frac{1}{\sqrt{(2a+1)(2b+1)\varepsilon_n}} \sin\left(\frac{(2k+1)\pi x}{2a+1}\right) \cos\left(\frac{2m\pi y}{2b+1}\right) .$$

The Green's function, g, can now be expressed in terms of these eigenfunctions as

$$g(x,y;x',y';s) = -\frac{C}{s}\left[G_{ee} + G_{oo} + G_{eo} + G_{oe}\right] , \tag{20}$$

where

$$
\begin{aligned}
G_{ee}(x,y;x',y';s) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{4\cosh\left(x'\sqrt{s_m^*}\right)\cosh\left(y'\sqrt{s_n^*}\right)\cosh\left(x\sqrt{s_m}\right)\cosh\left(y\sqrt{s_n}\right)}{\varepsilon_m\varepsilon_n(2a+1)(2b+1)(s_m+s_n-s)},\\
G_{oo}(x,y;x',y';s) &= \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\frac{4\sinh\left(x'\sqrt{s_k^*}\right)\sinh\left(y'\sqrt{s_\ell^*}\right)\sinh\left(x\sqrt{s_k}\right)\sinh\left(y\sqrt{s_\ell}\right)}{(2a+1)(2b+1)(s_k+s_\ell-s)},\\
G_{eo}(x,y;x',y';s) &= \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{4\cosh\left(x'\sqrt{s_m^*}\right)\sinh\left(y'\sqrt{s_\ell^*}\right)\cosh\left(x\sqrt{s_m}\right)\sinh\left(y\sqrt{s_\ell}\right)}{\varepsilon_m(2a+1)(2b+1)(s_m+s_\ell-s)},\\
G_{ee}(x,y;x',y';s) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{4\sinh\left(x'\sqrt{s_k^*}\right)\cosh\left(y'\sqrt{s_n^*}\right)\sinh\left(x\sqrt{s_k}\right)\cosh\left(y\sqrt{s_n}\right)}{\varepsilon_n(2a+1)(2b+1)(s_k+s_n-s)}.
\end{aligned}
\tag{21}
$$

Since each term of the summations has a simple pole, the inverse Laplace transformation is merely a matter of evaluating the corresponding residues. If a single oscillator at (x′,y′) is detuned at time zero by C locking ranges, there will be a double pole at s=0 leading to a term which is linear in time. This term exhibits the shift in the ensemble frequency due to the change in the average of the free running frequencies subsequent to the detuning of one oscillator. The overall dynamic solution in such a case is

$$
\begin{aligned}
\phi(x,y,x',y';\tau) &= \frac{C\tau}{(2a+1)(2b+1)}u(\tau)\\
&+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{4\cos\left(\frac{2m\pi x'}{2a+1}\right)\cos\left(\frac{2n\pi y'}{2b+1}\right)\cos\left(\frac{2m\pi x}{2a+1}\right)\cos\left(\frac{2n\pi y}{2b+1}\right)}{\varepsilon_m\varepsilon_n(2a+1)(2b+1)\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau)\\
&+\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\frac{4\sin\left(\frac{(2k+1)\pi x'}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y'}{2b+1}\right)\sin\left(\frac{(2k+1)\pi x}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y}{2b+1}\right)}{(2a+1)(2b+1)\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau)\\
&+\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{4\cos\left(\frac{2m\pi x'}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y'}{2b+1}\right)\cos\left(\frac{2m\pi x}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y}{2b+1}\right)}{\varepsilon_m(2a+1)(2b+1)\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{4\sin\left(\frac{(2k+1)\pi x'}{2a+1}\right)\cos\left(\frac{2n\pi y'}{2b+1}\right)\sin\left(\frac{(2k+1)\pi x}{2a+1}\right)\cos\left(\frac{2n\pi y}{2b+1}\right)}{\varepsilon_n(2a+1)(2b+1)\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau).
\end{aligned}
\tag{22}
$$

It should be noted at this point that the series in the above expression diverge to infinity at the location of the detuned oscillator. This is an artifact caused by the representation of the detuning distribution by a spatial Dirac delta function. A more appropriate representation is a square pulse covering one unit cell centered at the detuned oscillator. The phase distribution corresponding to this representation can be obtained from the above solution by integration over the unit cell. This multiplies each of the terms of the series by a sinc function of each of the summation indices. The solution then becomes

$$
\begin{aligned}
\phi(x,y;x',y';\tau) = {} & \frac{C\tau}{(2a+1)(2b+1)}u(\tau) \\
& + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{4\cos\left(\frac{2m\pi x'}{2a+1}\right)\cos\left(\frac{2n\pi y'}{2b+1}\right)\cos\left(\frac{2m\pi x}{2a+1}\right)\cos\left(\frac{2n\pi y}{2b+1}\right)}{\varepsilon_m\varepsilon_n(2a+1)(2b+1)\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau) \\
& \qquad \times \frac{\sin\left(\frac{m\pi}{2a+1}\right)}{\left(\frac{m\pi}{2a+1}\right)}\frac{\sin\left(\frac{n\pi}{2a+1}\right)}{\left(\frac{n\pi}{2a+1}\right)} \\
& + \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \frac{4\sin\left(\frac{(2k+1)\pi x'}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y'}{2b+1}\right)\sin\left(\frac{(2k+1)\pi x}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y}{2b+1}\right)}{(2a+1)(2b+1)\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau) \\
& \qquad \times \frac{\sin\left(\frac{(2k+1)\pi}{2(2a+1)}\right)}{\left(\frac{(2k+1\pi)}{2(2a+1)}\right)}\frac{\sin\left(\frac{(2\ell+1)\pi}{2(2a+1)}\right)}{\left(\frac{(2\ell+1)\pi}{2(2a+1)}\right)} \\
& + \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{4\cos\left(\frac{2m\pi x'}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y'}{2b+1}\right)\cos\left(\frac{2m\pi x}{2a+1}\right)\sin\left(\frac{(2\ell+1)\pi y}{2b+1}\right)}{\varepsilon_m(2a+1)(2b+1)\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{2m\pi}{2a+1}\right)^2+\left(\frac{(2\ell+1)\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau) \\
& \qquad \times \frac{\sin\left(\frac{m\pi}{2a+1}\right)}{\left(\frac{m\pi}{2a+1}\right)}\frac{\sin\left(\frac{(2\ell+1)\pi}{2(2a+1)}\right)}{\left(\frac{(2\ell+1)\pi}{2(2a+1)}\right)} \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{4\sin\left(\frac{(2k+1)\pi x'}{2a+1}\right)\cos\left(\frac{2n\pi y'}{2b+1}\right)\sin\left(\frac{(2k+1)\pi x}{2a+1}\right)\cos\left(\frac{2n\pi y}{2b+1}\right)}{\varepsilon_n(2a+1)(2b+1)\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]}\left\{1-e^{-\left[\left(\frac{(2k+1)\pi}{2a+1}\right)^2+\left(\frac{2n\pi}{2b+1}\right)^2\right]\tau}\right\}u(\tau) \\
& \qquad \times \frac{\sin\left(\frac{(2k+1)\pi}{2(2a+1)}\right)}{\left(\frac{(2k+1\pi)}{2(2a+1)}\right)}\frac{\sin\left(\frac{n\pi}{2a+1}\right)}{\left(\frac{n\pi}{2a+1}\right)}.
\end{aligned}
\tag{23}
$$

This solution, excluding the term linear in time, evaluated for a 21 by 21 array with the oscillator at (5,2) detuned by one locking range is shown in Figure 2 for a sequence of times beginning at one inverse locking range after the detuning and ending at infinite time.

From (22) or (23) one can infer that, in the general case, the response time of the array to detuning of one oscillator, which corresponds to the pole closest to the origin, is given by

$$\tau_0 = \frac{1}{\pi^2}\left[\frac{(2a+1)^2(2b+1)^2}{(2a+1)^2+(2b+1)^2}\right]. \tag{24}$$

If the detuned oscillator lies on the y axis (center line of the array), this result becomes

$$\tau_0 = \frac{1}{\pi^2}\left[\frac{(2a+1)^2(2b+1)^2}{(2a+1)^2+4(2b+1)^2}\right]. \tag{25}$$

Similarly, if the detuned oscillator lies on the x axis

$$\tau_0 = \frac{1}{\pi^2}\left[\frac{(2a+1)^2(2b+1)^2}{4(2a+1)^2+(2b+1)^2}\right]. \tag{26}$$

Finally, if one detunes the center oscillator

$$\tau_0 = \frac{1}{4\pi^2}\left[\frac{(2a+1)^2(2b+1)^2}{(2a+1)^2+(2b+1)^2}\right]. \tag{27}$$

All of these time constants are roughly proportional to the number of elements in the array.

IV. BEAMSTEERING IN TWO DIMENSIONS

Beamsteering requires that the aperture constant phase surface be planar but with normal tilted with respect to the normal to the aperture. The beam is directed normal to this tilted constant phase surface and is thus steered away from the aperture normal direction. To obtain such an aperture phase distribution, consider the following oscillator tuning function.

$$\frac{\omega_{tune}}{\Delta\omega_{lock}} = \frac{\omega_{ref}}{\Delta\omega_{lock}} + \left[\Omega_{x_1}\delta(x'-c_1)+\Omega_{x_2}\delta(x'-c_2)+\Omega_{y_1}\delta(y'-d_1)+\Omega_{y_2}\delta(y'-d_2)\right]u(\tau) \tag{28}$$

The dynamic aperture phase behavior resulting from this tuning can be obtained by integrating the source function corresponding to (28) (the quantity in square brackets) multiplied by the Green's function (20) over the area of the array. The result is

$$\phi(x,y;\tau) = \left(\frac{\Omega_{x_1} + \Omega_{x_2}}{2a+1} + \frac{\Omega_{y_1} + \Omega_{y_2}}{2b+1} \right) \tau u(\tau)$$

$$+ \frac{\Omega_{x_1}}{(2a+1)} \sum_{p=1}^{\infty} \frac{\cos\left[\frac{p\pi}{2a+1}(x-c_1)\right] + (-1)^p \cos\left[\frac{p\pi}{2a+1}(x+c_1)\right]}{\left(\frac{p\pi}{2a+1}\right)^2} \left[1 - e^{-\left(\frac{p\pi}{2a+1}\right)^2 \tau}\right] u(\tau)$$

$$+ \frac{\Omega_{x_2}}{(2a+1)} \sum_{p=1}^{\infty} \frac{\cos\left[\frac{p\pi}{2a+1}(x-c_2)\right] + (-1)^p \cos\left[\frac{p\pi}{2a+1}(x+c_2)\right]}{\left(\frac{p\pi}{2a+1}\right)^2} \left[1 - e^{-\left(\frac{p\pi}{2a+1}\right)^2 \tau}\right] u(\tau)$$

$$+ \frac{\Omega_{y_1}}{(2b+1)} \sum_{p=1}^{\infty} \frac{\cos\left[\frac{p\pi}{2b+1}(y-d_1)\right] + (-1)^p \cos\left[\frac{p\pi}{2b+1}(y+d_1)\right]}{\left(\frac{p\pi}{2b+1}\right)^2} \left[1 - e^{-\left(\frac{p\pi}{2b+1}\right)^2 \tau}\right] u(\tau)$$

$$+ \frac{\Omega_{y_2}}{(2b+1)} \sum_{p=1}^{\infty} \frac{\cos\left[\frac{p\pi}{2b+1}(y-d_2)\right] + (-1)^p \cos\left[\frac{p\pi}{2a+1}(y+d_2)\right]}{\left(\frac{p\pi}{2b+1}\right)^2} \left[1 - e^{-\left(\frac{p\pi}{2b+1}\right)^2 \tau}\right] u(\tau). \tag{29}$$

Note that the Dirac delta function representation of the detuning distribution has again been used here. Since no artifactual infinities arise in this case, it is not necessary to replace it by the unit pulse representation. The resulting difference in the solution would be nearly negligible in that it merely amounts to averaging each summation over a unit range of the independent variable (i.e., x or y as the case may be). However, the expressions for the steady state phase distribution would be somewhat more complicated. Therefore, for simplicity of presentation, the Dirac delta function representation is used.

In steady state, the exponentials in (29) are zero and the remaining summations can be evaluated in closed form to yield

$$
\begin{aligned}
\phi(x,y;\tau) = {} & \left(\frac{\Omega_{x_1} + \Omega_{x_2}}{2a+1} + \frac{\Omega_{y_1} + \Omega_{y_2}}{2b+1} \right) \tau \\
& + \Omega_{x_1} \left(\frac{2a+1}{2} \right) \left[\frac{1}{6} - \frac{|x - c_1|}{2a+1} + \frac{c_1^2 + x^2}{(2a+1)^2} \right] \\
& + \Omega_{x_2} \left(\frac{2a+1}{2} \right) \left[\frac{1}{6} - \frac{|x - c_2|}{2a+1} + \frac{c_2^2 + x^2}{(2a+1)^2} \right] \\
& + \Omega_{y_1} \left(\frac{2b+1}{2} \right) \left[\frac{1}{6} - \frac{|y - d_1|}{2b+1} + \frac{d_1^2 + y^2}{(2b+1)^2} \right] \\
& + \Omega_{y_2} \left(\frac{2b+1}{2} \right) \left[\frac{1}{6} - \frac{|y - d_2|}{2b+1} + \frac{d_2^2 + y^2}{(2b+1)^2} \right] .
\end{aligned} \tag{30}
$$

This expression can also be written in the form

$$
\begin{aligned}
\phi(x,y;\tau) = {} & \left(\frac{\Omega_{x_1} + \Omega_{x_2}}{2a+1} + \frac{\Omega_{y_1} + \Omega_{y_2}}{2b+1} \right) \tau \\
& + \left[\frac{\Omega_{x_2} + \Omega_{x_1}}{2} \right] \left[\frac{2a+1}{6} + \frac{1}{2} \left(\frac{c_2^2 + c_1^2}{2a+1} \right) - \frac{1}{2} \left(|x - c_2| + |x - c_1| \right) + \frac{x^2}{2a+1} \right] \\
& + \left[\frac{\Omega_{x_2} - \Omega_{x_1}}{2} \right] \left[\frac{1}{2} \left(\frac{c_2^2 - c_1^2}{2a+1} \right) + \frac{1}{2} \left(|x - c_2| - |x - c_1| \right) \right] \\
& + \left[\frac{\Omega_{y_2} + \Omega_{y_1}}{2} \right] \left[\frac{2b+1}{6} + \frac{1}{2} \left(\frac{d_2^2 + d_1^2}{2b+1} \right) - \frac{1}{2} \left(|y - d_2| + |y - d_1| \right) + \frac{y^2}{2b+1} \right] \\
& + \left[\frac{\Omega_{y_2} - \Omega_{y_1}}{2} \right] \left[\frac{1}{2} \left(\frac{d_2^2 - d_1^2}{2b+1} \right) + \frac{1}{2} \left(|y - d_2| - |y - d_1| \right) \right] ,
\end{aligned} \tag{31}
$$

which indicates more clearly that symmetric tuning gives rise to quadratic phase distributions while antisymmetric tuning leads to linear phase distributions. In fact, if the constants are chosen such that,

$$
\begin{aligned}
c_1 &= -c_2 = -c \quad , \\
d_1 &= -d_2 = -d \quad , \\
\Omega_{x_1} &= -\Omega_{x_2} = -\Omega_x \quad , \\
\Omega_{x_1} &= -\Omega_{x_2} = -\Omega_x \quad ,
\end{aligned} \tag{32}
$$

the steady state result reduces to

$$\phi(x,y;\tau) = \frac{\Omega_x}{2}\left(|x+c| - |x-c|\right) + \frac{\Omega_y}{2}\left(|y+d| - |y-d|\right), \tag{33}$$

which represents a tilted planar phase distribution appropriate for beamsteering wherein Ω_x controls the phase slope in the x direction and Ω_y controls the phase slope in the y direction. Finally, it is noted that these steady state solutions can be obtained by direct solution of the differential equation with the time derivative term set equal to zero. The procedure then becomes analogous to that used in electrostatics as described previously [3][4]. The detuning plays the role of electrostatic charge density and the phase plays the role of electrostatic potential.

Figure 3 illustrates the behavior of the aperture phase of a two-dimensional array under detuning of the perimeter oscillators in such a manner as to steer the beam. This is obtained from the above theory by setting c=a, d=b, and

$$\begin{aligned} \Omega_x &= 2\pi \frac{h}{\lambda} \sin\theta_0 \cos\phi_0 \\ \Omega_y &= 2\pi \frac{h}{\lambda} \sin\theta_0 \sin\phi_0 \end{aligned} \tag{34}$$

where h is the element spacing of the radiating aperture, λ is the free space wavelength, and coordinates (θ_0, ϕ_0) denote the desired beam direction in spherical coordinates with polar axis normal to the array. In the case shown, detuning appropriate to θ_0=30° and ϕ_0=-110° is applied at τ=0 and the sequence of plots shows the dynamic behavior of the aperture phase as it evolves toward the steady state distribution shown in the last plot of the sequence. Figure 4 shows this same steering transient in terms of the beam peaks, indicated by the dots, and the three dB contours, shown as the closed curves, as they evolve subsequent to detuning of the perimeter oscillators. Note that the beam shape changes early in the transient period due to the phase aberration arising across the aperture as the oscillators readjust to the new tuning. This results in a temporary reduction in gain as indicated in Figure 5 which shows the array gain as a function of time during the transient period. For typical GHz range oscillator locking ranges of tens of MHz, this transient period will be on the order of a few microseconds. Lastly, Figure 6 illustrates a sequence of four detunings applied in rapid succession and corresponding to four steered beam positions. Note that significantly greater beam shape distortion is evident during the transient period when the beam is steered from one off axis position to another compared with that arising when the beam is steered to or from the array normal direction.

IV. CONCLUDING REMARKS

Arrays of coupled electronic oscillators have been proposed as a means of controlling the aperture phase of a phased array antenna. A theoretical formalism developed in the context of one dimensional arrays has been extended in the present work to two-dimensional arrays and provides a convenient means of analyzing the phase dynamics of such arrays and the resulting radiated beam dynamics. Computational evaluations of the analytic solutions for phase and gain have been used to illustrate the transient behavior to be expected when moving the beam from one angle to another. The results indicate that the fundamental time constant of such an array is roughly proportional to the number of oscillators.

The fact that the phase difference between adjacent oscillators is limited to ninety degrees may seem to limit the beam steering to less than thirty degrees for half wavelength spacing. However, aside from the obvious method of decreasing the electrical spacing of the radiating elements [1], two possibilities exist for expanding this limit. One may radiate the second harmonic of the oscillator signal thus effectively doubling the available phase shift. Alternatively, one may double the phase shift by radiating the signal from only every other oscillator. Both techniques extend the scan range to endfire. The last option, of course, quadruples the required number of oscillators for a two-dimensional array.

It appears that the beamsteering approach treated here holds promise for greatly simplifying the required control system and for reducing the overall parts count in beamsteering arrays.

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Figure Captions

Figure 1. Boundary condition artifice for a two-dimensional oscillator array.

Figure 2. Phase dynamics for one oscillator step detuned at time zero.

Figure 3. Phase dynamics during beam steering.

Figure 4. Antenna beam peak and three dB contours.

Figure 5. Antenna gain during beam steering.

Figure 6. Sequential beam steering.

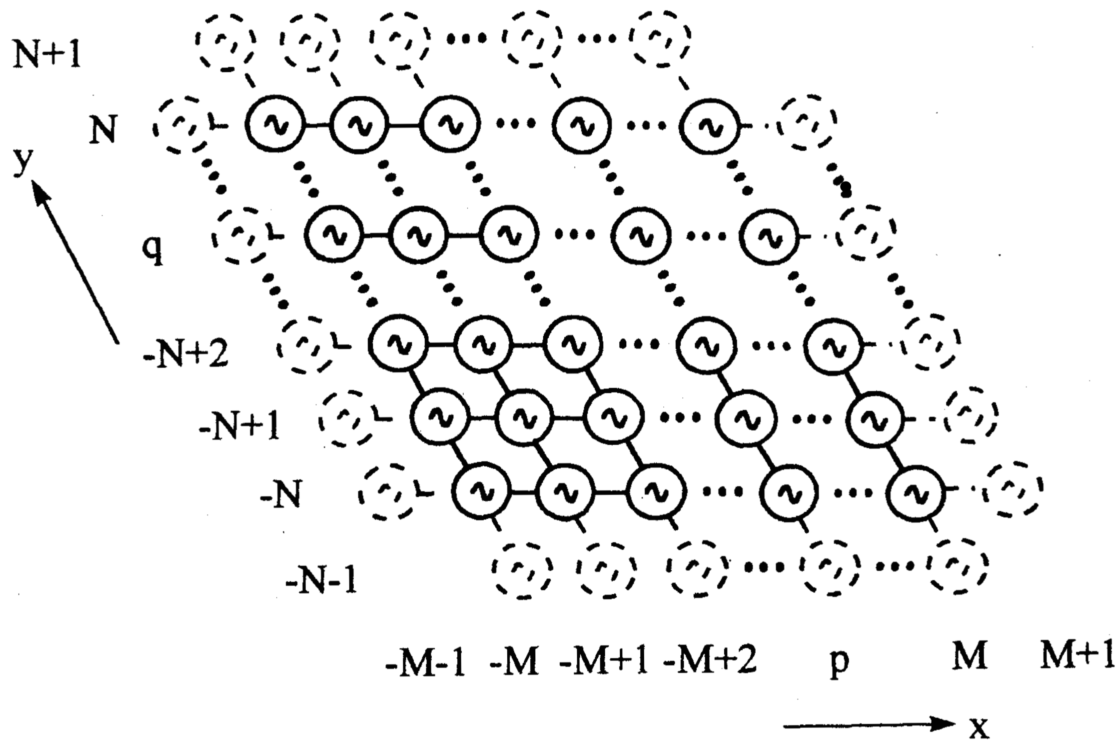

Figure 1. Boundary condition artifice for a two-dimensional oscillator array.

Oscillator Phases

Two Dimensional Array

Oscillator at (5,2) detuned one locking range.

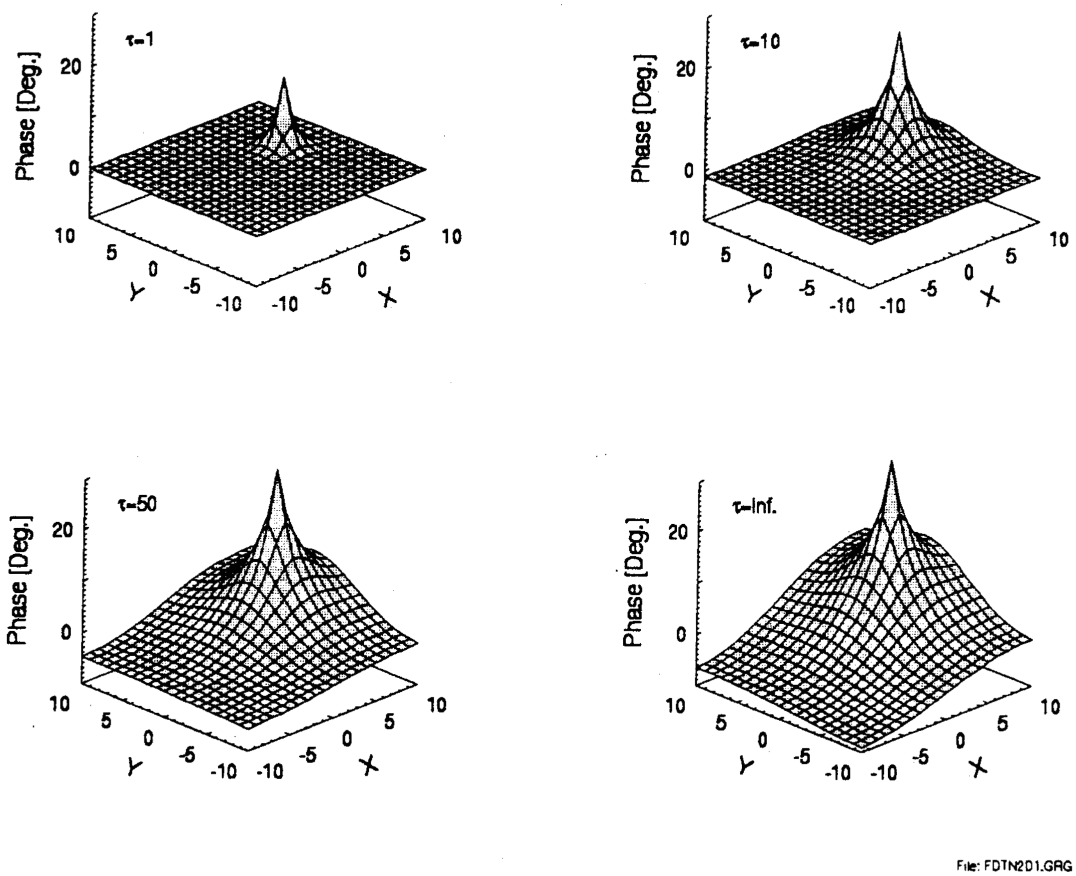

Figure 2. Phase dynamics for one oscillator step detuned at time zero.

Oscillator Phases

Two Dimensional Array

Edge oscillators detuned for beam steering.

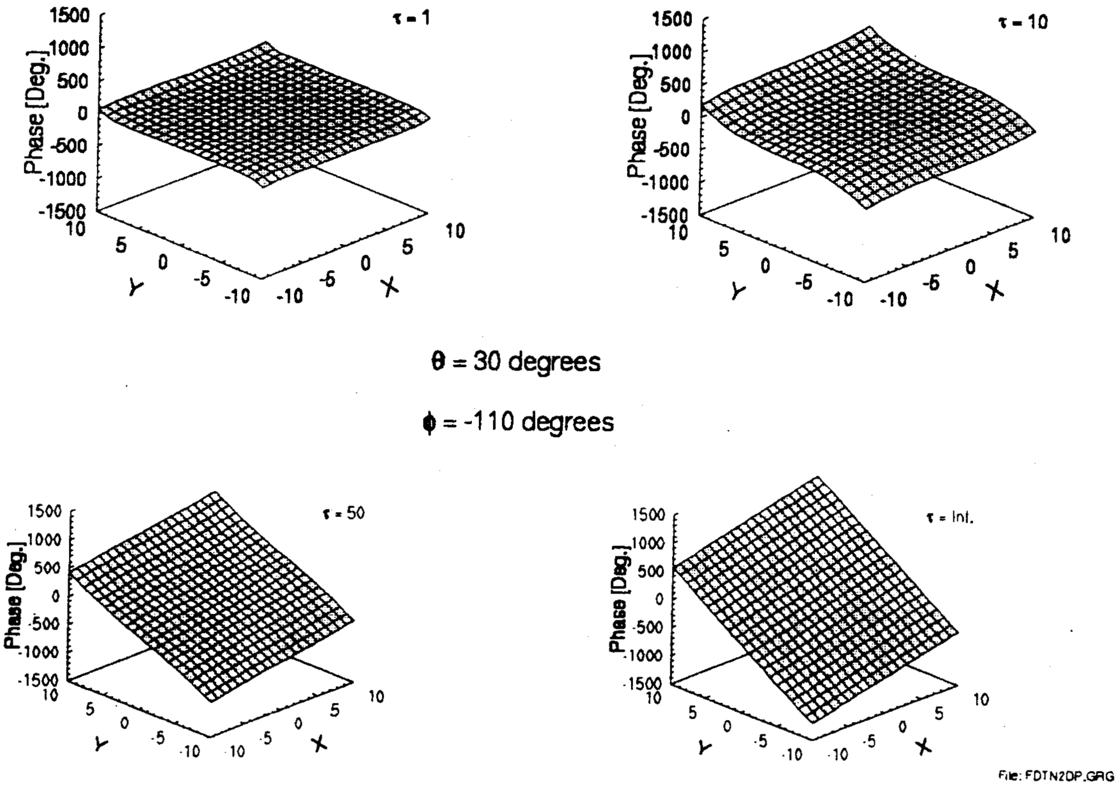

Figure 3. Phase dynamics during beam steering.

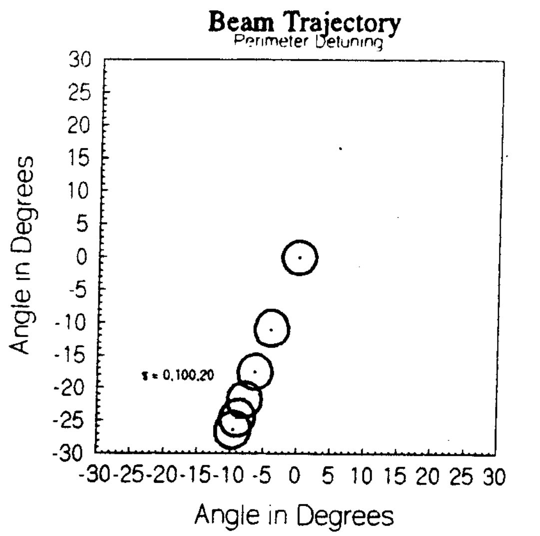

Figure 4. Antenna beam peak and three dB contours.

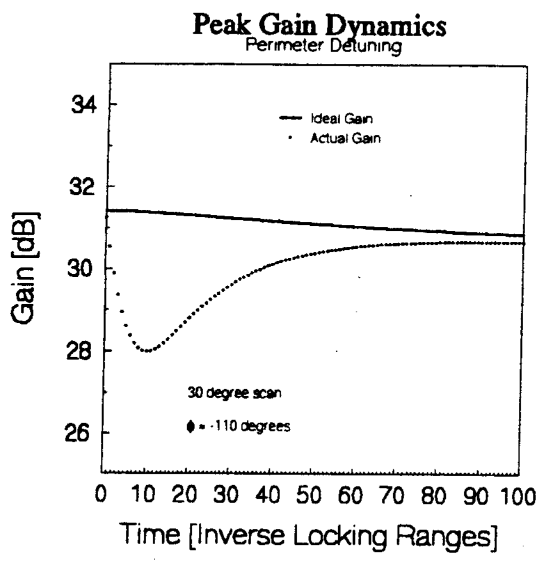

Figure 5. Antenna gain during beam steering.

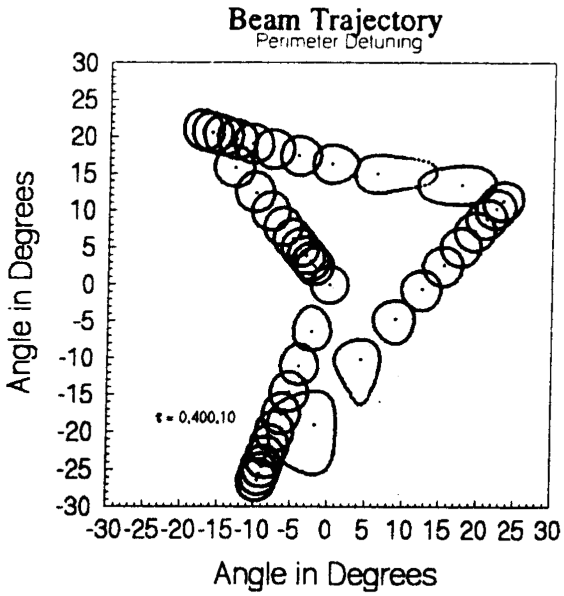

Figure 6. Sequential beam steering.